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Quantum resonances and ratchets in free-falling frames

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Quantum resonance (QR) is defined in the free-falling frame of the quantum kicked particle subjected to gravity. The general QR conditions are derived. They imply the rationality of the gravity parameter η, the kicking-period parameter $\tau/(2\pi)$, and the quasimomentum β. Exact results are obtained concerning wave-packet evolution for arbitrary periodic kicking potentials in the case of integer $\tau/(2\pi)$ (the main QRs). It is shown that a quantum ratchet generally arises in this case for resonant β. The noninertial nature of the free-falling frame affects the ratchet by effectively changing the kicking potential to one depending on (β,η). For a simple class of initial wave packets, it is explicitly shown that the ratchet characteristics are determined to a large extent by symmetry properties and by number-theoretical features of η.

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The quantum kicked particle in the presence of gravity has attracted much interest recently [1–7] following the experimental discovery of the "quantum accelerator modes" (QAMs) of freely falling atoms periodically kicked by pulses [1]. The QAMs were observed in the free-falling frame in a strong quantum regime and their explanation was at the focus of all the theoretical studies [4–7] which we briefly summarize here. Using dimensionless quantities and notation introduced in Ref. [4], the general Hamiltonian for the system is

$$\hat{H}=\frac{\hat{p}^2}{2}-\frac{\eta}{\tau}\hat{x}+kV(\hat{x})\sum_t \delta(t'-t\tau), \tag{1}$$

where $(\hat{p},\hat{x})$ are momentum and position operators, η is proportional to the gravity force (in the direction of the positive x axis), τ is the kicking period, k is a nonintegrability parameter, $V(x)$ is a periodic potential, and t' and t are the continuous and "integer" times. The units are chosen so that the particle mass is 1, $\hbar=1$, and the period of $V(x)$ is 2π; the standard potential $V(x)=\cos(x)$ was used in all works. The transformation to the free-falling frame is accomplished by applying the gauge transformation $\exp(i\eta\hat{x}t'/\tau)$ to the Schrödinger equation for (1) [4]. One then finds that the Hamiltonian in this frame is

$$\hat{H}_{\mathrm{f}}=\frac{(\hat{p}+\eta t'/\tau)^2}{2}+kV(\hat{x})\sum_t \delta(t'-t\tau). \tag{2}$$

Unlike (1), $\hat{H}_{\mathrm{f}}$ in Eq. (2) is translationally invariant in $\hat{x}$, implying the conservation of a quasimomentum β (the "fractional part" of $\hat{p}$, $0\leq\beta<1$) in the quantum evolution under $\hat{H}_{\mathrm{f}}$; at fixed β, one can consider x as an angle θ and write $\hat{p}=\hat{N}+\beta$, where $\hat{N}=-id/d\theta$ is an angular-momentum operator with integer eigenvalues n (see more details in note [8]). Then, (2) becomes the Hamiltonian of a kicked-rotor system in the free-falling frame. The one-period evolution operator, from $t'=t\tau+0$ to $t'=(t+1)\tau+0$, is given by

$$\hat{U}_{\beta}(t)=\exp[-ikV(\hat{\theta})]\exp[-i(\tau/2)(\hat{N}+\beta+\eta t+\eta/2)^2], \tag{3}$$

up to an irrelevant constant phase factor. Now, the value of $\tau=2\pi l_0$, where l_0 is a positive integer, corresponds to the main quantum resonances (QRs) of (3) in the absence of gravity ($\eta=0$) [9–11]. It was shown in Ref. [4] that $\tau=2\pi l_0+\epsilon$ defines, for sufficiently small $\epsilon\neq 0$ and for any η, a "quasiclassical" regime in which ϵ plays the role of a fictitious Planck's constant. In this regime, the quantum evolution under (3) can be approximately described by a classical map. Then, a wave packet initially trapped in an accelerator-mode island of this map "accelerates," i.e., the expectation value $\langle\hat{N}\rangle$ of $\hat{N}$ in the wave packet grows linearly in time; this is a QAM. The experimentally observed robustness of QAMs under variations of τ near $\tau=2\pi l_0$ was explained, in the framework of the quasiclassical approximation, as a "mode-locking" phenomenon [5–7] (see also the conclusion). Theoretical predictions were verified by several experiments [3,5].

In this paper, the system (3) with arbitrary periodic potential $V(\theta)$ is systematically approached in a different way. The concept of QR is introduced for this system ($\eta\neq 0$) and exact results are derived concerning its quantum-resonant dynamics. A consistent definition of QR for $\eta\neq 0$ requires the time-dependent operator (3) to be essentially periodic in t with some finite period T; QR can then be defined on the basis of the evolution operator in T kicks. The general conditions for QR, given by Eqs. (5)–(7) below, imply the rationality of η, $\tau/(2\pi)$, and β. Exact results for wave-packet evolution under (3) are obtained in the case of integer $\tau/(2\pi)$ (main QRs). We find that in this case the noninertial nature of the free-falling frame effectively changes $V(\theta)$ to a potential $V_{\beta,\eta}(\theta)$. We then show that a linear growth of $\langle\hat{N}\rangle$ in time generally occurs for resonant quasimomentum (7). This is a purely quantum "ratchet" effect [12,13], a directed current without a biased force, caused, e.g., by some asymmetry in the system. QR ratchets have been investigated recently [13] for the usual kicked rotor ($\beta=\eta=0$). We emphasize that for $\eta\neq 0$ there is *no* biased force in the system (2): gravity is classically not felt in the free-falling frame. In fact, (2) satisfies the conditions for a ratchet Hamiltonian [12] but the kinetic energy is time dependent, reflecting the noninertial nature of the free-falling frame. It is this time dependence that affects the QR ratchet through the effective potential $V_{\beta,\eta}(\theta)$. In particular, the ratchet current vanishes if $V_{\beta,\eta}(\theta)$ and the initial wave packet have a common point symmetry. For a simple class of initial wave packets, we derive closed explicit results for the linear-growth coefficient and we show that the ratchet characteristics are determined to a large extent by symmetry properties and by number-theoretical features of η.

QR in free-falling frames. QR is the quadratic growth of the kinetic-energy expectation value in time, due to the translational invariance of some basic evolution operator $\hat{U}$ for the system in phase-space; this invariance leads to a band quasienergy spectrum of $\hat{U}$ and thus to QR. A basic operator $\hat{U}$ for (3) can be consistently defined only if $\hat{U}_\beta(t+T)=\hat{U}_\beta(t)$ for some period T. Then $\hat{U}=\hat{U}_{\beta,T}(t)$, where

$$\hat{U}_{\beta,T}(t)=\hat{U}_\beta(t+T-1)\cdots\hat{U}_\beta(t+1)\hat{U}_\beta(t) \tag{4}$$

is the evolution operator in T kicks, and the operators $\hat{U}_{\beta,T}(t)$ for all t are equivalent (similar), due to $\hat{U}_{\beta,T}(t+1)=\hat{U}_\beta(t)\hat{U}_{\beta,T}(t)\hat{U}_\beta^{-1}(t)$. This allows one to associate with $\hat{U}_{\beta,T}(t)$ a meaningful (essentially t-independent) quasienergy problem. To derive explicit conditions for $\hat{U}_\beta(t+T)=\hat{U}_\beta(t)$, one must exploit the fact that $\hat{U}_\beta(t)$ is defined up to an arbitrary, physically irrelevant phase factor which may depend only on t. We thus replace $\hat{U}_\beta(t)$ by $\hat{U}'_\beta(t)=\exp(ia_1t+ia_2t^2)\hat{U}_\beta(t)$, where a_1 and a_2 are constants to be determined. Using Eq. (3) and the fact that $\hat{N}$ has integer eigenvalues, we easily get from $\hat{U}'_\beta(t+T)=\hat{U}'_\beta(t)$ that $a_1=\tau\eta(\beta+\eta/2)$, $a_2=\tau\eta^2/2$, and

$$\Omega\equiv\frac{\tau\eta}{2\pi}=\frac{w}{T}, \tag{5}$$

where w is some integer. Equation (5), i.e., the rationality of Ω, is the only physically relevant condition for the existence of a basic operator $\hat{U}_{\beta,T}(t)$. We shall assume that (w,T) are coprime, so that T is the smallest period for given rational value of Ω.

We now require $\hat{U}_{\beta,T}(t)$ to satisfy the basic QR condition for kicked-rotor systems [10,14,15], i.e., to be invariant under translations $\hat{T}_q=\exp(-iq\hat{\theta})$ by q (an integer) in the angular momentum $\hat{N}$: $[\hat{U}_{\beta,T}(t),\hat{T}_q]=0$. In the last relation, we can neglect, of course, any t-dependent phase factor attached to $\hat{U}_\beta(t)$ (see above) and just use the definition (4) of $\hat{U}_{\beta,T}(t)$ with $\hat{U}_\beta(t)$ given by Eq. (3). Using also Eq. (5) and, again, the fact that $\hat{N}$ has integer eigenvalues, we find after a straightforward calculation that $[\hat{U}_{\beta,T}(t),\hat{T}_q]=0$ implies that

$$\frac{\tau}{2\pi}=\frac{l}{q}, \tag{6}$$

$$\beta=\frac{r}{lT}-\frac{q}{2}-\frac{qw}{2l}\ \mathrm{mod}(1), \tag{7}$$

where l and r are integers. The QR conditions (6) and (7) can be analyzed as in the $\eta=0$ case [10]. Assuming, for definiteness and without loss of generality, that l and q are positive, we write $l=gl_0$ and $q=gq_0$, where l_0 and q_0 are coprime positive integers and g is the greatest common factor of (l,q). It is then clear that already at fixed $\tau/(2\pi)=l_0/q_0$ a resonant quasimomentum (7) can take *any* rational value β_r in $[0,1)$; this is because g can be always chosen so that $r=[\beta_r+gq_0/2+wq_0/(2l_0)]gl_0T$ is integer. For given $\beta=\beta_r$, we shall choose g as the smallest positive integer satisfying the latter requirement, so as to yield the minimal values of $l=gl_0$ and $q=gq_0$. We denote β_r by $\beta_{r,g}$, where the integer r above labels all the different values of β_r for given minimal g.

The quasienergy states $\phi(\theta)$ for $\beta=\beta_{r,g}$ are the simultaneous eigenstates of $\hat{U}_{\beta,T}(t)$ and $\hat{T}_q$: $\hat{U}_{\beta,T}(t)\phi(\theta)=\exp(-i\omega)\phi(\theta)$, $\hat{T}_q\phi(\theta)=\exp(-iq\alpha)\phi(\theta)$, where ω is the quasienergy and α is a "quasiangle," $0\leqslant\alpha<2\pi/q$. Using standard methods [14,15], it is easy to show that at fixed α one generally has q quasienergy levels $\omega_b(\alpha,\beta)$, $b=0,\ldots,q-1$; as α is varied continuously, these q levels typically "broaden" into q distinct bands (having nonzero width). This leads to QR, i.e., the asymptotic behavior $\langle\psi_{vT}|\hat{N}^2|\psi_{vT}\rangle\sim 2D(vT)^2$; here v is a large integer, $\psi_{vT}(\theta)=\hat{U}^v_{\beta,T}(0)\psi_0(\theta)$ is any evolving wave packet, and D is some coefficient.

Case of main QRs. From now on, we shall focus on the case of $\tau=2\pi l_0$ ($q_0=1$), the main QRs. The quantum evolution of wave packets under (3) can be exactly calculated in this case for arbitrary values of η and β, i.e., not just for the QR values determined by Eqs. (5) and (7). In fact, since $\hat{N}$ has integer eigenvalues, the relation $\exp(-i\pi l_0\hat{N}^2)=\exp(-i\pi l_0\hat{N})$ holds, so that Eq. (3) can be expressed for $\tau=2\pi l_0$ as follows:

$$\hat{U}_\beta(t)=\exp[-ikV(\hat{\theta})]\exp[-i(\tau_\beta+\pi l_0\eta+2\pi l_0\eta t)\hat{N}], \tag{8}$$

where $\tau_\beta=\pi l_0(2\beta+1)$ and an irrelevant phase factor has been neglected. We note that the second exponential operator in Eq. (8) is just a shift in θ. Thus the result of successive applications of (8) on an initial wave packet $\psi_0(\theta)$ can be written in a closed form:

$$\begin{aligned}\psi_t(\theta)&=\hat{U}_\beta(t-1)\cdots\hat{U}_\beta(1)\hat{U}_\beta(0)\psi_0(\theta)\\&=\exp[-ik\bar{V}_{\beta,\eta,t}(\theta)]\psi_0(\theta-\tau_\beta t-\pi l_0\eta t^2),\end{aligned} \tag{9}$$

where

$$\bar{V}_{\beta,\eta,t}(\theta)=\sum_{s=0}^{t-1}V(\theta-\tau_\beta s-2\pi l_0\eta ts+\pi l_0\eta s^2). \tag{10}$$

More explicit expressions for Eqs. (9) and (10) can be obtained for $\eta=w/(l_0T)$, i.e., the values of η corresponding to the main QRs ($\tau=2\pi l_0$) by Eq. (5). Let us leave β arbitrary for the moment and choose the time t in a natural way as a multiple v of the basic period T, $t=vT$. Then, writing $s=s'+v'T$, with $s'=0,\ldots,T-1$ and $v'=0,\ldots,v-1$, the sum in Eq. (10) can be decomposed into two sums over s' and v'. Using also the Fourier expansion

$$V(\theta) = \sum_m V_m \exp(-im\theta), \tag{11}$$

we find from Eqs. (9) and (10) with $\eta = w/(l_0 T)$ that

$$\psi_{vT}(\theta) = \exp[-ik\bar{V}_{\beta,\eta,vT}(\theta)]\psi_0(\theta - \tau_{\beta,w} vT), \tag{12}$$

where $\tau_{\beta,w} = \tau_\beta + \pi w = \pi(2l_0\beta + l_0 + w)$ and

$$\bar{V}_{\beta,\eta,vT}(\theta) = \sum_m V_m W_{m,\beta,\eta} \frac{\sin(m\tau_{\beta,w} vT/2)}{\sin(m\tau_{\beta,w} T/2)} \times e^{im(v-1)\tau_{\beta,w}T/2} \exp(-im\theta). \tag{13}$$

Here

$$W_{m,\beta,\eta} = \sum_{s=0}^{T-1} \exp[im(\tau_\beta s - \pi w s^2/T)] \tag{14}$$

is a "form factor" reflecting the noninertial nature of the free-falling frame, i.e., the time dependence of the kinetic energy in Eq. (2), in one period T. This factor, which is a generalized Gauss sum [6,16], effectively changes V_m in Eq. (13) to $V_m W_{m,\beta,\eta}$, which may be considered as the harmonics of a potential $V_{\beta,\eta}(\theta) = \Sigma_m V_m W_{m,\beta,\eta} \exp(-im\theta)$. For $T=1$, corresponding to $\eta = w/l_0$ (and, of course, also to $\eta = 0$), $W_{m,\beta,\eta} = 1$ and $V_{\beta,\eta}(\theta) = V(\theta)$. Then, the only effect of gravity on Eq. (12) is through the quantity $\tau_{\beta,w}$.

QR ratchets. The general QR behavior $\langle \psi_{vT} | \hat{N}^2 | \psi_{vT} \rangle \sim 2D(vT)^2$ for resonant $\beta = \beta_{r,g}$ (see above) suggests that a quantum-ratchet effect, i.e., a linear growth of $\langle \hat{N} \rangle_{vT} \equiv \langle \psi_{vT} | \hat{N} | \psi_{vT} \rangle$ under the evolution (12), may also occur for $\beta = \beta_{r,g}$ and sufficiently large v:

$$\langle \hat{N} \rangle_{vT} \approx \langle \hat{N} \rangle_0 + RvT, \tag{15}$$

where R is some nonzero coefficient. We now show that this is indeed the case for general potentials (11) and initial wave packets $\psi_0(\theta)$. At the same time, a formula for R is derived.

We start from the general expansion

$$|\psi_0(\theta)|^2 = \frac{1}{2\pi} \sum_m C(m) \exp(im\theta), \tag{16}$$

where $C(m) = \Sigma_n \tilde{\psi}_0(m+n)\tilde{\psi}_0^*(n)$ are correlations of the initial wave packet in its angular-momentum representation $\tilde{\psi}_0(n)$. Using Eqs. (12), (13), and (16), we get

$$\langle \hat{N} \rangle_{vT} = -i \int_0^{2\pi} d\theta \psi_{vT}^*(\theta) \frac{d\psi_{vT}(\theta)}{d\theta} = \langle \hat{N} \rangle_0 + ik \sum_{m \neq 0} m V_m W_{m,\beta,\eta} C(m) \times \frac{\sin(m\tau_{\beta,w} vT/2)}{\sin(m\tau_{\beta,w} T/2)} e^{-im(v+1)\tau_{\beta,w}T/2}, \tag{17}$$

where normalization of $\psi_0(\theta)$ is assumed, $\int_0^{2\pi} |\psi_0(\theta)|^2 d\theta = 1$. Now, a linear growth of (17) in v can arise only if $m\tau_{\beta,w}T/2 = r_m\pi$ for some $m \neq 0$, where r_m is integer; then, the contribution of the last three terms in which T appears in Eq. (17) is just equal to v. Using $\tau_{\beta,w} = \pi(2l_0\beta + l_0 + w)$ in $m\tau_{\beta,w}T/2 = r_m\pi$, we find that β must satisfy

$$\beta = \frac{r_m}{ml_0T} - \frac{1}{2} - \frac{w}{2l_0} \bmod(1). \tag{18}$$

By comparing Eq. (18) with Eq. (7), in which $l = gl_0$ and $q = gq_0 = g$ for some "minimal" g (see above), we see that Eq. (18) gives just a resonant value $\beta_{r,g}$ of β: m is some multiple of g ($m = jg$, j integer) and $r_m = j[r + l_0Tg(1-g)/2]$ for some integer r. Then, by collecting all the terms with $m = jg$ in Eq. (17), we obtain a formula for the coefficient R in Eq. (15):

$$R = -\frac{2kg}{T} \sum_{j>0} j \,\mathrm{Im}[V_{jg} W_{jg,\beta,\eta} C(jg)]. \tag{19}$$

Thus for given resonant quasimomentum $\beta = \beta_{r,g}$, $R \neq 0$ only if there exist sufficiently high harmonics V_m and correlations $C(m)$, with $m = jg$, and the sum of the corresponding terms in Eq. (19) is nonzero. These conditions are satisfied by general $V(\theta)$ and $\psi_0(\theta)$. A very simple case of $R = 0$ is when $V_m W_{m,\beta,\eta} C(m)$ is real for all m. This occurs, e.g., when the system is "symmetric," i.e., when both the effective potential $V_{\beta,\eta}(\theta) = \Sigma_m V_m W_{m,\beta,\eta} \exp(-im\theta)$ and $\psi_0(\theta)$ have a point symmetry around the *same* center, say $\theta = 0$: $V_{\beta,\eta}(-\theta) = V_{\beta,\eta}(\theta)$ and $\psi_0(-\theta) = \pm\psi_0(\theta)$ (inversion) or $\psi_0^*(-\theta) = \pm\psi_0(\theta)$ (inversion with time reversal); this implies that $V_m W_{m,\beta,\eta}$ and $C(m)$ [see Eq. (16)] are both real. We emphasize that the QR quadratic behavior of $\langle \psi_{vT} | \hat{N}^2 | \psi_{vT} \rangle$ is usually *not* affected by such symmetries (see example below).

As an illustration, we consider the simple class of initial wave packets $\psi_0(\theta) = F[1 + A\exp(-i\theta)]$, where A is some complex constant and $F = [2\pi(1+|A|^2)]^{-1/2}$ is a normalization factor. Writing $A = |A|\exp(i\gamma)$, we see that $\psi_0(\theta)$ has a symmetry center at $\theta = \gamma$, i.e., $\psi_0^*(2\gamma - \theta) = \psi_0(\theta)$. The only nonzero correlations in Eq. (16) are $C(0) = 1$, $C(1) = 2\pi A^* F^2$, and $C(-1) = C^*(1)$. Thus $R \neq 0$ in Eq. (19) only for resonant quasimomenta $\beta = \beta_{r,g}$ with $g = 1$; one has $R = -(2k/T)\mathrm{Im}[V_1 W_{1,\beta,\eta} C(1)]$, so that no essential generality is lost by choosing $V(\theta) = \cos(\theta)$ from now on. To obtain a more explicit expression for R, one has to evaluate $W_{1,\beta,\eta}$. Let us assume, for simplicity, that w is positive and odd and $l_0 = 1$, so that $l = gl_0 = 1$, $q = gq_0 = 1$ and, from Eq. (7), $\beta = \beta_{r,1} = r/T$, $r = 0, \ldots, T-1$. For convenience, the latter set of β values will be arranged in a different order, $\beta = \beta_{r,1} = rw/T \bmod(1)$ [recall that (w, T) are coprime]. Then, the form factor (14) for $m = 1$ and $\tau_\beta = \pi l_0(2\beta + 1) = 2\pi rw/T + \pi$ can be exactly calculated using known results about Gauss sums [17]. We find that $W_{1,\beta,\eta} = \sqrt{T}J \exp(i\gamma_{\beta,\eta})$, where the values of J and the phase $\gamma_{\beta,\eta}$ are listed in Table I for three different cases of T.

In Table I, $\left(\frac{a}{b}\right)$ denotes the Jacobi symbol [18], so that $J = \pm 1$. The effective potential is $V_{\beta,\eta}(\theta) = \sqrt{T}J\cos(\theta - \gamma_{\beta,\eta})$ and has a symmetry center at $\theta = \gamma_{\beta,\eta}$, i.e., $V_{\beta,\eta}(2\gamma_{\beta,\eta} - \theta) = V_{\beta,\eta}(\theta)$. We obtain from all the results above:

TABLE I. Values of J and $\gamma_{\beta,\eta}$ for $\beta=rw/T \bmod(1)$, $r=0,\ldots,T-1$.

Case	J	$\gamma_{\beta,\eta}$
T even	$\sqrt{2}\cos(\pi w/4)\left(\frac{2T}{w}\right)$	$\pi[r+wr^2/T+w(T-1)/4]$
$T \bmod(4)=1$	$\left(\frac{2w}{T}\right)$	$\pi(r+wr^2/T)$
$T \bmod(4)=3$	$\left(\frac{2w}{T}\right)$	$\pi(r+wr^2/T+1/2)$

$$R = \frac{k|A|J}{\sqrt{T}(1+|A|^2)} \sin(\gamma - \gamma_{\beta,\eta}). \tag{20}$$

We thus see from Eq. (20) that the noninertial nature of the free-falling frame causes a phase shift by $\gamma_{\beta,\eta}$ and a suppression of R by a factor of $\sqrt{T}$, relative to the case when gravity is absent (with $T=J=1$ and $\gamma_{\beta,\eta}=0$). At fixed $|A|$, k, and T, $|R|$ is completely determined by the distance $\Delta\gamma=|\gamma-\gamma_{\beta,\eta}|$ between the symmetry centers of $\psi_0(\theta)$ and $V_{\beta,\eta}(\theta)$. For $\Delta\gamma=0$, these centers coincide and $R=0$; $|R|$ is largest for $\Delta\gamma=\pi/2$, a value which may be viewed as corresponding to a "maximal asymmetry" situation. The ratchet-current direction is always given by the sign of $J\sin(\gamma-\gamma_{\beta,\eta})$, where $J=\pm 1$ depends entirely on number-theoretical features of (w,T) (see Table I). The symmetry properties do not affect the QR quadratic behavior $\langle\psi_{vT}|\hat{N}^2|\psi_{vT}\rangle \sim 2D(vT)^2$. In fact, using $\langle\psi_{vT}|\hat{N}^2|\psi_{vT}\rangle=\int_0^{2\pi}d\theta|d\psi_{vT}(\theta)/d\theta|^2$ and Eqs. (12) and (13), we easily find that $D=k^2/(4T)$, independent of $(\gamma,\gamma_{\beta,\eta})$.

In conclusion, QR can be consistently defined for the system (3) provided the rationality condition (5) is satisfied. It should be noted that $\Omega=\tau\eta/(2\pi)$ in Eq. (5) is one of the two parameters featured by the classical map which approximates (3) in the quasiclassical regime of $\tau=2\pi l_0+\epsilon$ [4–7] (see also the Introduction); the second parameter is a nonintegrability one, $\tilde{k}=k|\epsilon|$. For sufficiently small $\tilde{k}$, there exist accelerator-mode islands whose winding number ν is "locked" to the value w/T for all Ω in a small interval around $\Omega=w/T$. Wave packets initially trapped in these islands lead to the QAMs, i.e., a linear growth of $\langle\hat{N}\rangle_{vT}\approx avT$, where $a\approx 2\pi(w/T-\Omega)/\epsilon$ [4]. For $\Omega=w/T$, $a=0$, but in the main-QR limit of $\epsilon\to 0$ the quasiclassical approximation must be replaced by the exact description of (3) given by the operator (8), with an exponent linear in $\hat{N}$. Such an evolution operator corresponds to an integrable classical map [19], in contrast with the nonintegrable quasiclassical map for $\epsilon\neq 0$, and generally gives a ratchet behavior. Thus while both a QR ratchet (15) with $\Omega=w/T$ and a QAM with $\nu=w/T$ exhibit a linear growth of $\langle\hat{N}\rangle_{vT}$, they are basically different in nature. However, one may systematically study the quasiclassical regime by using the approach introduced in this paper, namely by considering at fixed η high-order QR ratchets with rational values of $\tau/(2\pi)$ in the vicinity of integers.

It is interesting to notice that QR ratchets arise, as we have shown, even for *symmetric* potentials and wave packets, when their symmetry centers do not coincide. Using methods similar to those for $\eta=0$ [10,11], it is easy to extend our fixed-β results to the general time evolution of the kicked particle, involving a superposition of the time evolutions for all β. One then finds that the kicked particle generally exhibits no ratchet current for $\tau=2\pi l_0$. The QR quadratic behavior for $\eta=0$ is known to be robust, under small variations of τ, on some initial time interval [9]. We expect a similar robustness of the $\eta\neq 0$ quantum-resonant evolution under small variations of η and $\tau/(2\pi)$ around their rational values. Our results should then be realizable in high-precision experiments such as recent ones [20] concerning $\eta=0$ QRs.

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[8] Since Eq. (2) is translationally invariant in $\hat{x}$, it can be restricted to the space of Bloch functions $\varphi_\beta(x)=\exp(i\beta x)\psi_\beta(x)$ with fixed β and arbitrary 2π-periodic function $\psi_\beta(x)$. In fact, $\hat{H}_{\rm f}\varphi_\beta(x)=\varphi'_\beta(x)=\exp(i\beta x)\psi'_\beta(x)$, where $\psi'_\beta(x)=\exp(i\beta x)\hat{H}_{{\rm f},\beta}\psi_\beta(x)$ and $\hat{H}_{{\rm f},\beta}$ is given by Eq. (2) with $\hat{p}$ replaced by $\hat{p}+\beta$. One can interpret x in 2π-periodic functions [such as $V(x)$, $\psi_\beta(x)$, and $\psi'_\beta(x)$] as an angle θ and $\hat{p}$ in $\hat{H}_{{\rm f},\beta}$ as an angular-momentum operator $\hat{N}=-id/d\theta$.

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[17] See, e.g., Chap. 1 in Ref. [16], in particular Sec. 1.5.

[18] The Jacobi symbol $\left(\frac{a}{b}\right)$, where a and b are integers and b is positive and odd, is first defined in the case that $b=l$, a prime number: If l divides a $(l>1)$, $\left(\frac{a}{l}\right)=0$; otherwise, $\left(\frac{a}{l}\right)=1$ if there exists an integer d such that l divides $a-d^2$ and $\left(\frac{a}{l}\right)=-1$ if such an integer does not exist. Then, if b is a product of L prime numbers not necessarily distinct, $b=l_1l_2\cdots l_L$, one defines $\left(\frac{a}{b}\right)=\left(\frac{a}{l_1}\right)\left(\frac{a}{l_2}\right)\cdots\left(\frac{a}{l_L}\right)$.

[19] M. V. Berry, Physica D **10**, 369 (1984).

[20] C. Ryu *et al.*, Phys. Rev. Lett. **96**, 160403 (2006).